Master thesis project *Uncertainty quantification in Stochastic Differential Equations and applications to Neurosciences*

Mention: Research

Advisors:

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Phd forseen: A PhD could be forseen, the subject is conceived so.

Project objectives:

Many mathematical models involve input parameters, which are not precisely known. Global sensitivity analysis aims to identify the parameters whose uncertainty has the largest impact on the variability of a quantity of interest - for instance by computing Sobol' sensitivity indices. In this project we consider stochastic models described by stochastic differential equations (SDE), whose coefficients depend on some uncertainty parameter $\xi \in \mathbb{R}^d$. More specifically, we are interested in harmonic oscillators perturbed with a gaussian white noise. More precisely, we consider a vector of uncertain parameters $\xi = (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d$ and the process $\big(Z_t := (X_t, Y_t) \in \mathbb{R}^2\,,\ t \geq 0\big)$ governed by the following Ito stochastic differential equation:

$$\begin{cases} dX_t &= Y_t dt \\ dY_t &= \sigma(\xi) dW_t - (c(\xi, X_t, Y_t) Y_t + \nabla V(\xi, X_t)) dt\,. \end{cases}$$

We assume that the process is ergodic with a unique invariant probability measure μ, and that the rate of convergence in the ergodic theorem is exponential. For such oscillators, we aim at studying the influence of the uncertain parameters $\xi_1, \ldots, \xi_d$ on quantities of interest defined from the density of the invariant probability measure μ, namely $(x, y) \mapsto p(x, y, \xi)$. We will consider as a toy model the Kramer oscillator with $\xi = (\sigma, \kappa, \alpha, \beta) \in \big(\mathbb{R}_+^*\big)^4$, $\sigma(\xi) = \sigma$, $c(\xi, X_t, Y_t) = \kappa Y_t$ and with the Duffing's potential $V(\xi, x) = \alpha x^4/4 - \beta x^2/2$. One knows that function p satisfies the stationary Fokker-Planck equation:

$$\mathcal{L}^* p(\cdot, \cdot, \xi) = 0, \quad \text{with constraints} \quad p \geq 0, \text{ and } \int_{\mathbb{R}} p(x, y, \xi) dx dy = 1,$$

where $\mathcal{L}^*$ is the formal adjoint of the infinitesimal generator associated to $\{(X_t, Y_t)\,,\ t \geq 0\}$.

Sensitivity analysis for models driven by systems of stochastic differential equations were presented, e.g., in [LMK15, EPPL20]. The main challenges of the present project are the hypoellipticity of the diffusion under study and the specific nature of the quantities of interest for sensitivity analysis.

The main steps to tackle these challenges will consist in:

- deriving a numerical scheme for the kinetic Fokker-Planck equation,
- proposing a metamodel for $\xi \mapsto p_n(\cdot, \cdot, \xi)$ numerical solution of the kinetic Fokker-Planck equation (based, e.g., on tools presented in [N09]),
- computing from evaluations of this metamodel sensitivity indices for different quantities of interest.

The final aim of this project is to handle the hypoelliptic Fitzhugh-Nagumo model arising from neurosciences (see for instance [LS18]). The quantities of interest for this study are defined from the density function $(x, y) \mapsto p(x, y, \xi)$ solution of the kinetic Fokker-Planck equation. In particular, the quantity $\int_0^{+\infty} y p(u, y, \xi) dy$ approximates the spike rate (subject to uncertainty ξ) for a specific range of the threshold u. It might also be interesting to consider directly as quantity of interest the density function

$y \mapsto p(u,y,\xi)/\int p(u,v,\xi)dv$ by computing sensitivity measures based on the Wasserstein metric, recently introduced in [FKL20].

Required competences

We seek for a student in probability and statistics with some knowledge and/or interest in PDE issues. Some knowledge of scientific computing is required (C/C++, Matlab, R or Python).

References

[LS18] Jose R. León and Adeline Samson, *Hypoelliptic stochastic FitzHugh-Nagumo neuronal model: Mixing, up-crossing and estimation of the spike rate*, The Annals of Applied Probability, Vol. 28, No. 4, 2018, pp 2243-2274.

[EPPL20] Pierre Etoré, Clémentine Prieur, Dang Khoi Pham and Long Li, *Global sensitivity analysis for models described by stochastic differential equations*, Methodology and Computing in Applied Probability, Vol. 22, 2020, pp 803-831.

[FKL20] Jean-Claude Fort, Thierry Klein and Agnès Lagnoux, *Global sensitivity analysis and Wasserstein spaces*, 2020, `https://arxiv.org/pdf/2007.12378.pdf`

[LMK15] O.P. Le Maître, and O.M. Knio. *PC analysis of stochastic differential equations driven by Wiener noise*, Reliability Engineering and System Safety, 2015, pp 107-124.

[N09] Anthony Nouy, *Recent developments in spectral stochastic methods for the numerical solution of stochastic partial differential equations*, Archives of Computational Methods in Engineering, Vol. 16, No. 3, 2009, pp 251-285.