Algorithms for Computing Exact Sum of Squares Decompositions

Research Context. Certification and validation of computational results is a major issue for modern sciences raising challenging problems at the interplay of mathematics and computational aspects of computer science. One can emphasize in this context several applications arising in the design of modern cyber-physical systems with a crucial need of *exact* certification. In particular, one tries to avoid incidents such as the Patriot missile crash in 1991, the FDIV Pentium bug in 1994 or more recently the collision of Google's self-driving car in 2016. These issues give rise to many mathematical problems. Polynomial optimization (which consists in computing the infimum of a polynomial function under algebraic constraints) is one of the most important and challenging one. The emergence of this exciting new field goes back to the last decade and has led to striking developments from a cross fertilization between (real) algebraic geometry, applied mathematics, theoretical computer science and engineering.

Goals Consider for instance the problem of minimizing $4x^4 + 4x^3y - 7x^2y^2 - 2xy^3 + 10y^4$ over $\mathbb{R}^2$. One way to certify that its minimum is 0 is to decompose this polynomial as a *sum of squares* (SOS), which is the core subject of study in real algebra. Here the decomposition is $(2xy + y^2)^2 + (2x^2 + xy - 3y^2)^2 \geq 0$. In general, one can compute such SOS decompositions by solving a *semidefinite program* (SDP) [2], which is a standard tool in applied mathematics and convex optimization. In SDP, one optimizes a linear function under the constraint that a given matrix is semidefinite positive, i.e. has only nonnegative eigenvalues. One particular issue arising while relying on SDP solvers is that they are often implemented using floating-point arithmetic, thus output only *approximate* certificates. The challenging goal of this internship is to design algorithms to compute *exact* certificates while controlling the bit complexity of the algorithmic procedures.

Preliminary work will consist of studying the exisiting algorithms to obtain exact SOS decompositions of nonnegative polynomials. In particular, the case of univariate polynomials has been recently handled in [3] by means of classical techniques from symbolic computation (root isolation, square-free decomposition). A promising research track would be to apply the certification algorithms from [3] to a multivariate polynomial through a reduction to the univariate case. The idea is to characterize the set of minimizers of this polynomial by exploiting the information given by the Jacobian, in the same spirit as in [4]. After designing the certification framework, further efforts should lead to provide the related bit complexity estimates, both on runtime and output size. Practical experiments shall be performed through implementing a tool within a computer algebra software such as Maple. One expected goal is to compare the performance of the tool with existing frameworks, e.g. the rationalization scheme (rounding and projection algorithm) developed in [1].

Working Context The internship will be co-advised by Victor Magron (CNRS) and Mohab Safey El Din (UPMC, PolSys INRIA team). The Master student will be hosted by the PolSys team in the LIP6 laboratory, located at Jussieu, in the heart of Paris (5-th district). PolSys is a joint project-team between INRIA, CNRS and University Pierre and Marie Curie. The group, led by Jean-Charles Faugère, is internationally recognized as one of the leading group in the area of solving systems of polynomial equations/inequalities using exact methods. It is used to welcome students in a nice working atmosphere.
Suitable for MSCI/STAT/DS/MI students. Motivated candidates should have a solid background in **either** optimization, real algebraic geometry or computer algebra. Good programming skills are also required. The candidates are kindly asked to send an e-mail with "M2 candidate" in the title, a CV and motivation letter to `victor.magron@polsys.lip6.fr` and `mohab.safey@lip6.fr`. pre-requisite.

A related PhD topic can be foreseen.

References

[1] E. Kaltofen, B. Li, Z. Yang, and L. Zhi. Exact Certification in Global Polynomial Optimization Via Sums-Of-Squares of Rational Functions with Rational Coefficients. *JSC*, 47(1):1–15, jan 2012.

[2] J.-B. Lasserre. Global Optimization with Polynomials and the Problem of Moments. *SIAM Journal on Optimization*, 11(3):796–817, 2001.

[3] V. Magron, M. Safey El Din, and M. Schweighofer. Algorithms for Sums of Squares Decompositions of Nonnegative Univariate Polynomials, 2017. Submitted.

[4] J. Nie. An exact jacobian sdp relaxation for polynomial optimization. *Mathematical Programming*, 137(1):225–255, Feb 2013.